Central Series, Commutator Series, Nilpotent Groups, and Solvable Groups

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1 Introduction

Given a group G, we are familiar with its *composition series*; that is, a series

$$G = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_{n-1} \lhd G_n = \{e\},$$

where, for all $0 \leq i \leq n-1$, the quotient G_i/G_{i+1} is simple. This series gives us a helpful characterization of solvable groups. However, there exists many other series decompositions of groups that give rise to useful characterizations. In this paper, we analyze three such series: the *upper central series*, the *lower central series*, and the *commutator series.* These series will allow us to define and classify *nilpotent groups*, which are a fundamental type of group between abelian and solavble groups. Nilpotent groups are also used heavily in the classification of Lie groups.

In Section 2, we define the Upper Central Series. In Section 3, we define nilpotent groups and provide some helpful examples. In Section 4, we prove our first classification theorem of nilpotent groups:

Theorem 1.1. *Let G be a finite group, and let $p_1, p_2, \ldots, p_n$ be all the distinct primes dividing its order. For each $1 \leq i \leq n$, let P_i be a Sylow p_i-subgroup of G. Then, the following statements are equivalent:*

1. *G is nilpotent.*
2. *If H is a proper subgroup of G, then H is a proper subgroup of $N_G(H)$.*
3. *Every Sylow subgroup is normal in G.*
4. *$G \cong P_1 \times P_2 \times \cdots \times P_n$.*

In Section 5, we prove our second classification theorem of nilpotent groups by proving *Frattini's Argument*, an important group-theoretic lemma:

Theorem 1.2. *A finite group is nilpotent if and only if every maximal subgroup is normal.*

In Section 6, we define the lower central series and investigate its relation to the upper central series. Finally, in Section 7, we define the commutator series and use it to prove an alternative characterization of solvable groups:

Theorem 1.3. *A group G is solvable if and only if $G^{(n)} = \{e\}$ for some $n \geq 0$.*

2 The Upper Central Series

The first series we examine is the *upper central series*, which will allow us to rigorously define nilpotent groups and prove a powerful classification theorem regarding them.

Definition 2.1 (Upper Central Series)**.** Let G be a group. We define the following sequence $Z_n(G)$ (where $n \in \mathbb{Z}_{\geq 0}$) of subgroups of G inductively:

- $Z_0(G) = \{e\}$
- Otherwise, $Z_{n+1}(G)$ is the subgroup of G containing $Z_n(G)$ such that

$$Z_{n+1}(G)/Z_n(G) \cong Z(G/Z_i(G)).$$

Then, the infinite series of subgroups

$$Z_0(G) \leq Z_1(G) \leq \cdots$$

is the *upper central series* of G.

From just the definition, it may be unclear that such a series exists; in particular, we would require that all the $Z_i(G)$ be normal subgroups of G due to the $Z(G/Z_i(G))$ term.

Lemma 2.1. *Given an upper central series $Z_0(G) \leq Z_1(G) \leq \cdots$ of a group G, $Z_n(G) \rhd G$ for all $n \in \mathbb{N}$.*

Proof. We induct on n, with the base case of $n = 0$ being clear since $Z_0(G) = \{e\}$ is clearly a normal subgroup of G. Now, assume $Z_n(G) \rhd G$ for some integer $n \geq 0$. By the Third and Fourth Isomorphism Theorems, since $Z_n(G) \rhd G$, there is a bijection between subgroups of G that contain $Z_n(G)$ and subgroups of $G/Z_n(G)$ which sends a subgroup $Z_n(G) \subseteq K \subseteq G$ to $\pi(K) \subseteq G/Z_n(G)$, where $\pi : G \to G/Z_n(G)$ is the standard quotient map. In particular, since $Z_{n+1}(G)$ contains $Z_n(G)$, this bijection sends $Z_{n+1}(G)$ to $Z_{n+1}(G)/Z_n(G)$. This is isomorphic to $Z(G/Z_n(G))$, which is a normal subgroup in $G/Z_n(G)$. By the Third and Fourth Isomorphism Theorems, since

$$Z_{n+1}(G)/Z_n(G) \cong Z(G/Z_n(G)) \rhd G/Z_n(G),$$

we have $Z_{n+1}(G) \rhd G$, as desired. ∎

Remark 2.1. For finite groups, this subgroups in the upper central series must eventually terminate in size since the order of subgroups cannot grow forever.

3 Nilpotent Groups

Definition 3.1 (Nilpotent Group)**.** Let $Z_0(G) \subseteq Z_1(G) \subseteq \cdots$ be the upper central series of a group G. If $Z_n(G) = G$ for some $n \in \mathbb{N}$, then G is *nilpotent*. If G is nilpotent, then the smallest n with $Z_n(G) = G$ is the *nilpotence class* of G.

Remark 3.1. Nilpotent groups can be thought of as a "stepping stone" between abelian groups and solvable groups, in that all abelian groups are nilpotent and all nilpotent groups are solvable. However, there exist nilpotent groups that are nonabelian (e.g. the dihedral groups, as shown in Lemma **??**) and solvable groups that are not nilpotent (e.g. S_3).

Remark 3.2. To see that every nilpotent group is solvable, note that the upper central series itself is a composition series, since the $Z_i(G) \triangleright Z_{i+1}(G)$ by definition and their quotient is abelian since it is defined as the center of a group.

We now examine several examples of nilpotent groups to elucidate this definition.

Example 3.1. Any abelian group is nilpotent with nilpotence class 1 (excluding the trivial group, which has nilpotence class 0). This is because, if G is abelian, $G = Z(G)$, so

$$Z_1(G)/Z_0(G) \cong Z_1(G) \cong Z(G/Z_0(G)) \cong Z(G) \cong G.$$

Lemma 3.1. *Let G be a p-group with order p^n for $n \in \mathbb{N}$. Then, G is nilpotent. Furthermore, if $n \geq 2$, then G has nilpotence class of at most $n-1$.*

Proof. Any subgroup of G is a p-group. In particular, this implies that, for all i, if $G/Z_i(G)$ is nontrivial, then so is $Z(G/Z_i(G))$. Therefore, if $Z_i(G) \neq G$, then $|Z(G/Z_i(G))| \geq p$ since it is a subgroup of $G/Z_i(G)$, which is a nontrivial p-group. This implies $|Z_{i+1}(G)| \geq p|Z_i(G)|$ when $Z_i(G) \neq G$. Applying this reasoning inductively, we get that, for each i,

$$|Z_i(G)| \geq \max(p^i, |G|).$$

This implies that

$$|Z_n(G)| \geq \max(p^n, |G|) = |G|,$$

so $|Z_n(G)| = |G|$ and G has nilpotence class at most n.

If $n \geq 2$, then we must show that G does not have nilpotence class n. If G did have nilpotence class n, then we would need $|Z_i(G)| = p^i$ for all $0 \leq i < n$. Since $n \geq 2$, we can consider $Z_{n-2}(G)$, which satisfies

$$|Z_n(G)/Z_{n-2}(G)| = p^2$$

- it is well known that a group of order p^2 is abelian. However, this would imply that

$$Z_{n-1}(G)/Z_{n-2}(G) \cong Z(G/Z_{n-2}(G)) \cong G/Z_{n-2}(G),$$

which would imply $Z_{n-1}(G) \cong G$, which is a contradiction. Thus, if $n \geq 2$, then G has nilpotence class of at most $n-1$, as desired. ■

Example 3.2. For an example of a group that is not nilpotent, consider the symmetric group S_3. We have $Z(S_3) = \{e\}$, so $Z_1(S_3) = \{e\}$. Repeating this logic. $Z_i(S_3) = \{e\}$ for all $i \in \mathbb{Z}_{\geq 0}$ and S_3 is not nilpotent. More generally, the center of S_n is trivial for all $n \geq 3$, so S_n is not nilpotent for $n \geq 3$ by the same logic.

4 Classifying Nilpotent Groups

Now that we are familiar with nilpotent groups, we now want to characterize them in ways that are easier to work with. The following powerful theorem provides many equivalent characterizations of finite nilpotent groups.

Theorem 4.1. *Let G be a finite group, and let $p_1, p_2, \ldots, p_n$ be all the distinct primes dividing its order. For each $1 \leq i \leq n$, let P_i be a Sylow p_i-subgroup of G. Then, the following statements are equivalent:*

1. *G is nilpotent.*

2. If H is a proper subgroup of G, then H is a proper subgroup of $N_G(H)$.

3. Every Sylow subgroup is normal in G.

4. $G \cong P_1 \times P_2 \times \cdots \times P_n$.

We defer the proof that (1) implies (2) to Section 6.

Before we begin the proof that (2) implies (3), we will define a *characteristic subgroup.*

Definition 4.1. A subgroup H of G is *characteristic* if every automorphism of G fixes H.

Remark 4.1. If a subgroup is characteristic, then it is normal since all conjugations are automorphisms.

Proof That (2) Implies (3). To show that (2) implies (3), we first prove a few lemmas:

Lemma 4.2. *If G is a group and p is a prime dividing $|G|$ such that there is a unique Sylow p-subgroup P of G, then P is characteristic in G.*

Proof. Any automorphism of G must map a Sylow p-subgroup to another Sylow p-subgroup. Since P is the unique Sylow p-subgroup P of G, it is fixed under any automorphism and is hence characteristic. ■

Lemma 4.3. *If $K \subseteq H \subseteq G$ is a chain of groups such that K is characteristic in H and H is normal in G, then K is normal in G.*

Proof. Since H is normal in G, it is fixed by conjugation in G. Since K is characteristic in H, any automorphism of H fixes K. Thus, conjugation in G fixes H, which then fixes K, implying K is normal in G, as desired. ■

Now, let P be a Sylow p_i-subgroup of G for some $1 \leq i \leq n$. We clearly have $P \rhd N_G(P)$. This implies P is the unique Sylow p_i-subgroup of $N_G(P)$, since all Sylow p-subgroups are conjugate by Sylow's Second Theorem. By Lemma 4.2, P is characteristic in $N_G(P)$. Since $N_G(P)$ is normal in $N_G(N_G(P))$, by Lemma 4.3, we have $P \lhd N_G(N_G(P))$. This implies $N_G(N_G(P)) \subseteq N_G(P)$, so $N_G(N_G(P)) = N_G(P)$. Then, by (2), we have $N_G(P) = G$, so $P \rhd G$, as desired. ■

Proof That (3) Implies (4). We now show that (3) implies (4). We first prove the following:

Lemma 4.4. *Let H and K be two subgroups of a group G. Then,*

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

Proof. Let $hk = h'k'$, for $h, h' \in H$ and $k, k' \in K$. Then, $h'^{-1}h = k'k^{-1}$, so $s \in H \cap K$, where $h'^{-1}h = k'k^{-1} = s$. This implies $k' = sk$ and $h' = hs^{-1}$. In other words, each distinct element of HK can be expressed as a product hk in at most $H \cap K$ ways. On the other hand, if $t \in H \cap K$, then for fixed $h \in H$ and $k \in K$, $ht^{-1} \in H$ and $tk \in K$, so

$$hk = (ht^{-1})(tk) \in HK,$$

meaning each distinct element of HK can be expressed as a product hk in at least $H \cap K$ ways. Combining these two statements, gives that each element of HK is expressed by exactly $H \cap K$ products. There are $|H||K|$ products in total, so

$$|HK| = \frac{|H||K|}{|H \cap K|},$$

as desired. ■

Since $P_1, P_2, \ldots, P_n$ are all distinct Sylow p_i-subgroups of G, they have pairwise trivial intersection. Furthermore, the product of any subset of them has trivial intersection with any subgroup not in the product. Applying Lemma 4.4 repeatedly, we get

$$|P_1P_2| = \frac{|P_1||P_2|}{|P_1 \cap P_2|} = |P_1||P_2|, |P_1P_2P_2| = \frac{|P_1P_2||P_3|}{|P_1P_2 \cap P_3|} = |P_1||P_2||P_3|,$$

and so on. Continuing inductively gives

$$|P_1P_2 \cdots P_n| = |P_1||P_2| \cdots |P_n| = |G|,$$

since $P_1, P_2, \ldots, P_n$ are Sylow p_i-subgroups for all the distinct primes dividing $|G|$.

Lemma 4.5. *If $H_1, H_2, \ldots, H_n$ are normal subgroups of a group G, then $H_1H_2 \cdots H_n$ is a subgroup of G.*

Proof. We prove the following subclaim:

Claim 4.6. *Let H and K be subgroups of a group G, with K normal. Then, HK is a subgroup of G.*

Proof. Let $h_1, h_2 \in H$ and $k_1, k_2 \in K$. We want to show that $(h_1k_1)(h_2k_2) \in HK$. We have

$$(h_1k_1)(h_2k_2) = h_1h_2(h_2^{-1}k_1h_2)k_2 \in HK,$$

where the last inclusion is due to the fact that K is normal, so $h_2^{-1}k_1h_2 \in K$. ■

We now apply Claim 4.6 inductively to prove the lemma. Indeed, applying the claim to H_1 and H_2 tells us that H_1H_2 is a subgroup of G. Since H_3 is normal in G, applying the claim again gives $H_1H_2H_3$ is a subgroup of G. Continuing like this for all n subgroups gives the desired lemma. ■

By Lemma 4.5, since we assume all Sylow subgroups are normal in G, $P_1P_2 \cdots P_n$ is a subgroup of G. Since $|P_1P_2 \cdots P_n| = |G|$, we must have $P_1P_2 \cdots P_n = G$. The upshot of this is that we now only have to show that

$$P_1P_2 \cdots P_n \cdots P_1 \times P_2 \times \cdots \times P_n.$$

We will prove this by induction - in particular, we will show that, for all $1 \leq i \leq n$,

$$P_1P_2 \cdots P_i \cong P_1 \times P_2 \times \cdots \times P_i.$$

The base case of $i = 1$ is clear. Now, assume the claim is true for some $1 \leq i \leq n-1$. Since

$$P_1P_2 \cdots P_i \cong P_1 \times P_2 \times \cdots \times P_i,$$

we have $|P_1P_2 \cdots P_i| = |P_1||P_2| \cdots |P_i|$. This is coprime to $|P_{i+1}|$. Since $P_{i+1} \cap P_1P_2 \cdots P_i$ is a subgroup of both groups, by Lagrange's Theorem, its order must divide both their orders. Thus, $P_{i+1} \cap P_1P_2 \cdots P_i$ must be trivial. It now remains to show that this implies

$$P_1P_2 \cdots P_iP_{i+1} \cong P_1 \times P_2 \times \cdots \times P_i \times P_{i+1}.$$

■

Proof That (4) Implies (1). We first prove the following two lemmas:

Lemma 4.7. *If $G_1, G_2, \ldots, G_n$ are groups, then*

$$Z(G_1 \times G_2 \times \cdots \times G_n) \cong Z(G_1) \times Z(G_2) \times \cdots \times Z(G_n).$$

Proof. Note that $(g_1, g_2, \ldots, g_n) \in Z(G_1 \times G_2 \times \cdots \times G_n)$ if, for all $(h_1, h_2, \ldots, h_n) \in G_1 \times G_2 \times \cdots \times G_n$, we have

$$\begin{aligned}(h_1, h_2, \ldots, h_n)(g_1, g_2, \ldots, g_n) = (h_1g_1, h_2g_2, \ldots, h_ng_n) &= (g_1h_1, g_2h_2, \ldots, g_nh_n) \\ &= (g_1, g_2, \ldots, g_n)(h_1, h_2, \ldots, h_n).\end{aligned}$$

Thus, we need $h_ig_i = g_ih_i$ for all $1 \leq i \leq n$, or equivalently, $g_i \in Z(G_i)$ for all $1 \leq i \leq n$. Thus, $Z(G_1 \times G_2 \times \cdots \times G_n) \subseteq Z(G_1) \times Z(G_2) \times \cdots \times Z(G_n)$.

On the other hand, if $(z_1, z_2, \ldots, z_n) \in Z(G_1) \times Z(G_2) \times \cdots \times Z(G_n)$, then, for any $(h_1, h_2, \ldots, h_n) \in G_1 \times G_2 \times \cdots \times G_n$, we have $h_iz_i = z_ih_i$ for all $1 \leq i \leq n$, so

$$(h_1, h_2, \ldots, h_n)(z_1, z_2, \ldots, z_n) = (z_1, z_2, \ldots, z_n)(h_1, h_2, \ldots, h_n).$$

Thus, $Z(G_1) \times Z(G_2) \times \cdots \times Z(G_n) \subseteq Z(G_1 \times G_2 \times \cdots \times G_n)$, so the two groups are equal, as desired. ∎

Lemma 4.8. *If $G = A_1 \times A_2 \times \cdots \times A_n$ and $B_1, B_2, \ldots, B_n$ are groups such that B_i is normal in A_i for all $1 \leq i \leq n$. Then, $B_1 \times B_2 \times \cdots \times B_n$ is normal in $A_1 \times \cdots \times A_n$ and*

$$(A_1 \times \cdots \times A_n)/(B_1 \times \cdots \times B_n) \cong (A_1/B_1) \times (A_2/B_2) \times \cdots \times (A_n/B_n).$$

Proof. We first show that $B_1 \times B_2 \times \cdots \times B_n$ is normal in $A_1 \times A_2 \times \cdots \times A_n$. Let $(b_1, b_2, \ldots, b_n) \in B_1 \times B_2 \times \cdots \times B_n$ and $(a_1, a_2, \ldots, a_n) \in A_1 \times A_2 \times \cdots \times A_n$. Then,

$$(a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n)(a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1}) = (a_1b_1a^{-1}, a_2b_2a_2^{-1}, \ldots, a_nb_na_n^{-1}) \in B_1 \times B_2 \times \cdots \times B_n,$$

where the last step is since each B_i is normal in A_i, so $a_ib_ia_i^{-1} \in B_i$ for all $1 \leq i \leq n$.

To show the second part of the lemma, consider the map

$$\phi : (A_1 \times A_2 \times \cdots \times A_n)/(B_1 \times B_2 \times \cdots \times B_n) \to (A_1/B_1) \times (A_2/B_2) \times \cdots \times (A_n/B_n)$$

that sends a coset $(a_1, a_2, \ldots, a_n)(B_1 \times \cdots \times B_n)$ to the tuple of cosets $(a_1B_1, a_2B_2, \ldots, a_nB_n)$. This map is well-defined and can be checked to be a homomorphism. Furthermore, its kernel is the identity coset, so it is injective. Since the domain and codomain have the same size, this homomorphism is then as isomorphism, which finishes. ∎

To prove that (4) implies (1), we strong induct on $|G|$, with the base case of $|G| = 1$ being vacuously true. Since each P_i is a nontrivial p-group, $Z(P_i)$ is nontrivial for all $1 \leq i \leq n$. Thus, $Z(G)$ is nontrivial and $|G/Z(G)| < |G|$. By Lemma 4.7 and Lemma 4.8, the hypotheses of (4) hold for $G/Z(G)$. Since $|G/Z(G)| < |G|$, $G/Z(G)$ is nilpotent by the inductive hypothesis.

Lemma 4.9. *Given a group G, if $G/Z(G)$ is nilpotent, then G is nilpotent.*

Proof. By the Third and Fourth Isomorphism Theorems, there is a bijection between subgroups of $G/Z(G)$ and subgroups of G that contain $Z(G)$. In particular, this means that the terms of the upper central series for $G/Z(G)$ biject to the upper central series for G beyond the first term. For example, the second term of the upper central series of G is $Z(G)$, which is the image of $\{e\}$ in the upper central series of $G/Z(G)$. Thus, the upper central series of G eventaully reaches G since $G/Z(G)$ is nilpotent, as desired. ∎

By Lemma 4.9, since $G/Z(G)$ is nilpotent, G is nilpotent, as desired. ∎

5 Another Classification of Nilpotent Groups

In this section, we prove another classification theorem for nilpotent groups:

Definition 5.1 (Maximal Subgroup)**.** A proper subgroup M of G is *maximal* if there is no subgroup H such that $M \subset H \subset G$.

Theorem 5.1. *A finite group is nilpotent if and only if every maximal subgroup is normal.*

To prove this, we utilize *Frattini's Argument*, a lemma that turns up in many group-theoretic contexts.

Lemma 5.2 (Frattini's Argument)**.** *Let G be a finite group, N be a normal subgroup of G, and P be a Sylow p-subgroup of N. Then, $G = NN_G(P)$.*

Proof. Fix some $g \in G$; we now want to show that there exist $n \in N$ and $p \in N_G(P)$ such that $g = np$. Let $P' = gPg^{-1}$. Since N is normal and $P \subseteq N$, $P' \subseteq N$. Furthermore, since P' and P are of the same order, P' is a Sylow p-subgroup of N. By Sylow's Second Theorem, P' and P are conjugate in N, so there exists some $n \in N$ such that $nPn^{-1} = P'$. Thus,

$$n^{-1}P'n = n^{-1}gPg^{-1}n = P,$$

which implies $n^{-1}g \in N_G(P)$. Let $p = n^{-1}g$. Then, $g = np$, where $n \in N$ and $p \in N_G(P)$. Thus, $G = NN_G(P)$, as desired. ■

We can now prove Theorem 5.1.

Proof of Theorem 5.1. In one direction, let G be some finite nilpotent group and M be a maximal subgroup of G. By Theorem 4.1, M is a proper subgroup of $N_G(M)$, but since it is maximal, we must have $N_G(M) = G$, so $M \rhd G$.

For the other direction, assume every maximal subgroup of G is normal. Let P be some Sylow p-subgroup of G, for some $p \mid |G|$. (If G is trivial, the statement is vacuously true, so we can assume otherwise). If $P \not\rhd G$, then $N_G(P)$ is a proper subgroup of G. Let M be a maximal subgroup of G containing $N_G(P)$. We have $M \rhd G$, so by Frattini's Argument, $G = MN_G(P)$. However, since $N_G(P) \subseteq M$, $MN_G(P) = M$, which contradicts the fact that M is a proper subgroup of G. Thus, $P \rhd G$. Since all Sylow subgroups are normal in G, by Theorem 4.1, G is nilpotent, as desired. ■

6 The Lower Central Series

There exists an analogue to the upper central series called the *lower central series* that can be used to classify nilpotent groups in exactly the same fashion as upper central series. This series is based on the notion of a *commutator*, which we define below.

Definition 6.1 (Commutator)**.** Given a group G and two elements $x, y \in G$, their *commutator*, notated $[x, y]$, is defined by

$$[x, y] = x^{-1}y^{-1}xy.$$

Extending this notion, if H, K are two subgroups of G, then their commutator is

$$[H, K] = \{[h, k] : h \in H, k \in K\}.$$

Remark 6.1. If $x, y \in G$ commute, then $[x, y] = e$.

Definition 6.2 (Lower Central Series)**.** Given a group G, we define the sequence of subgroups G^n of G inductively, for $n \in \mathbb{Z}_{\geq 0}$:

- $G^0 = G$
- Otherwise, $G^{n+1} = [G, G^n]$.

The chain of subgroups

$$G = G^0 \supseteq G^1 \supseteq \cdots$$

is the *lower central series* of G.

Remark 6.2. I haven't finished this section yet, but it will eventually include the rest of the proof of the first classification theorem of nilpotent groups as well as a proof that the definition of nilpotnet groups for lower central series is equivalent to the one for upper central series.

7 The Commutator Series

The final series we will examine is the *commutator series*, which is very similar to the lower central series:

Definition 7.1. Given a group G, define the following sequence of subgroups $G^{(n)}$ of G inductively, for $n \in \mathbb{Z}_{\geq 0}$.

- $G^{(0)} = G$
- Otherwise, $G^{(n+1)} = [G^{(n)}, G^{(n)}]$.

Then, the chain of subgroups

$$G^{(0)} \supseteq G^{(1)} \supseteq \cdots$$

is the *commutator series*, or *derived series* of G.

We can use the commutator series to display another characterization of solvable groups.

Theorem 7.1. *A group G is solvable if and only if $G^{(n)} = \{e\}$ for some $n \geq 0$.*

Proof. We first prove the following lemma:

Lemma 7.2. *Let G be a group and let $G' = [G, G]$. Then, $G' \rhd G$ and G/G' is abelian. Furthermore, if H is some normal subgroup of G such that G/H is abelian, then $G' \subseteq H$.*

Proof. To show G' is normal, note that, for $g, g_1, g_2 \in G$,

$$g[g_1, g_2]g^{-1} = gg_1^{-1}g_2^{-1}g_1g_2g^{-1} = gg_1^{-1}g^{-1}gg_2^{-1}g^{-1}gg_1`g^{-1}gg_2g^{-1} = [gg_1g^{-1}, gg_2g^{-1}],$$

so it is closed under conjugation and hence normal.

To show that G/G' is abelian, let xG' and yG' be elements of G/G'. Then, we have

$$(xG')(yG') = (xy)G' = (yxx^{-1}y^{-1}xy)G' = (yx[x, y])G' = (yx)G' = (yG')(xG'),$$

as desired.

For the last part of the lemma, let $H \rhd G$ and G/H be abelian. Let $xH, yH \in G/H$. Then, we have $(xH)(yH) = (yH)(xH)$, which is equivalent to

$$H = (xH)^{-1}(yH)^{-1}(xH)(yH) = (x^{-1}y^{-1}xy)H = [x, y]H.$$

Thus, $[x, y] \in H$ for all $x, y \in G$, meaning $G' \subseteq H$, as desired. ∎

Now, assume that G is solvable, so it has a composition series

$$\{e\} = G_0 \lhd G_1 \lhd \cdots \lhd G_k = G$$

with each composition factor G_{i+1}/G_i abelian. We will prove by induction that $G^{(i)} \subseteq G_{k-i}$ for all $0 \leq i \leq n$. This is clearly true for $i = 0$, so assume $G^{(i)} \subseteq G_{k-i}$ for some $0 \leq i \leq n-1$. Then, we have

$$G^{(i+1)} = [G^{(i)}, G^{(i)}] \subseteq [G_{k-i}, G_{k-i}].$$

By Lemma 7.2, since G_{k-i}/G_{k-i-1} is abelian, we have $[G_{k-i}, G_{k-i}] \subseteq G_{k-i-1}$, so $G^{(i+1)} \subseteq G_{k-1-i}$, as desired. This completes the induction, which implies $G^{(k)} = 1$, as desired.

On the other hand, if $G^{(n)} = \{e\}$ for some $n \geq 0$, then the composition series

$$\{e\} = G^{(n)} \rhd G^{(n-1)} \rhd \cdots \rhd G^{(0)} = G$$

is a composition series for G, so it is solvable. ■